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Saturation and Inverse Theorems for Certain Operators in C or L_p, $1 \leqslant p < \infty$

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0. INTRODUCTION

Let K_{np} be a generalization of the Jackson kernel, namely,

$$K_{np}(t) = \lambda_{n'}^{-1}\{\sin(n't/2)/\sin(t/2)\}^{2p},$$

where $n' = [n/p] + 1$ (here $[x]$ is the largest integer $\leqslant x$), and the constant $\lambda_{n'}$ is chosen so that

$$\int_{-\pi}^{\pi} K_{np}(t)\, dt = 1, \qquad n, p = 1, 2,...$$

[1]. We put

$$k_{np}(u) = K_{[n/2]p}(t), \qquad u = \sin(t/2),$$

$$\rho_n^{-1} = \int_{-1}^{1} k_{np}(u)\, du = (1/2)\int_{-\pi}^{\pi} \cos(t/2)\, K_{[n/2]p}(t)\, dt \approx 1. \tag{0.1}$$

Then $\rho_n^{-1}k_{np}$ is even, non-negative, and a polynomial in u. Let $C[I]$ be the class of all continuous real functions on I, and let Π_n be the subclass of $C[I]$ consisting of all algebraic polynomials of degree n or less. If $I = [a, b] \subset (0, 1)$, we consider a certain operator with the kernel $\rho_n^{-1}k_{np}$, which maps $C[I]$ into Π_n. To this end we extend $f \in C[I]$ to a function $F \in C[R]$, where $R = (-\infty, \infty)$, which satisfies the following conditions (1)–(4):

(1) It is 2-periodic.

(2) It is even.

(3) Let $\delta_0 = (1/2)\min\{a, 1 - b\}$ and $\delta_r = \delta_0/(r + 1)$ $(r = 1, 2,...)$. Then $F(x) \equiv 0$ for $x \in [0, 2\delta_0 - \delta_r] \cup [1 - 2\delta_0 + \delta_r, 1]$.

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(4) F is sufficiently smooth, for instance, if $f^{(k)} \in \mathrm{Lip}(1; C[a, b])$, then $F^{(k)} \in \mathrm{Lip}(1; C[0, 1])$, or

$$w_r(F; h) \leqslant M_{rf} w_r(f; h), \qquad 0 < h \leqslant (b-a)/r, \tag{0.2}$$

where $w_r(g; \cdot)$ is the rth modulus of smoothness of g, and M_{rf} is a constant depending on r and f.

In this case we put

$$\begin{aligned} I_{npr}(F; x) &= \rho_n \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \left(\frac{1}{j}\right) \int_0^j F(u)\, k_{np}((x-u)/j)\, du \\ &= \rho_n \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \int_{(-x)/j}^{(j-x)/j} F(x+ju)\, k_{np}(u)\, du, \end{aligned} \tag{0.3}$$

where p is the smallest integer $\geqslant (r+2)/2$, that is,

$$2p - 1 = \begin{cases} r+2 & \text{if } r \text{ is odd} \\ r+1 & \text{if } r \text{ is even.} \end{cases}$$

We can determine the saturation class of the operator I_{npr} (Corollary 2.1). However, our methods are also applicable to other kinds of operators, for example, those of Korovkin type [2–4]. Let φ be a nonnegative, even, and continuous function on $[-c, c]$, decreasing on $[0, c]$ and such that $\varphi(0) = 1$ and $0 \leqslant \varphi(t) < 1$ for $0 < t \leqslant c$. In this case we define

$$k_n(u) = \varphi^n(u), \qquad n = 1, 2, \ldots . \tag{0.4}$$

Let $f \in C[I]$, where $I = [a, b] \subset (0, c)$. By the same method as (0.2) we extend f to $F \in C[R]$ which is $2c$-periodic and even. For such a function F we define the operator

$$\begin{aligned} K_{nr}(F; x) &= \rho_n \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \left(\frac{1}{j}\right) \int_0^{jc} F(u)\, k_n((x-u)/j)\, du \\ &= \rho_n \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \int_{(-x)/j}^{(jc-x)/j} F(x+ju)\, k_n(u)\, du \end{aligned} \tag{0.5}$$

where

$$\rho_n^{-1} = \int_{-c}^{c} k_n(u)\, du, \qquad n = 1, 2, \ldots .$$

For simplicity we consider only the case $c = 1$.

In this paper we consider a certain class of operators which contains the above, and we determine the saturation class of such operators. Let k_n be even, positive, and belonging to $C^r(0, 1]$, and let $I = [a, b] \subset (0, 1)$. For each $f \in C[I]$ we consider a function $F \in C[R]$ which is extended by method (0.2). Then we define a linear operator

$$K_{nr}(F; x) = \rho_n \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \left(\frac{1}{j}\right) \int_0^j F(u)\, k_n((x-u)/j)\, du$$

$$= \rho_n \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \int_{(-x)/j}^{(j-x)/j} F(x+ju)\, k_n(u)\, du, \qquad (0.6)$$

where

$$\rho_n^{-1} = \int_{-1}^{1} k_n(u)\, du, \qquad n = 1, 2, \ldots,$$

which maps $C[I]$ into itself. For this operator $K_{nr}(F)$ we make the following assumptions: There is a positive number λ such that

(I) $\rho_n \int_\delta^1 k_n(u)\, du = \rho_n \int_{-1}^{-\delta} k_n(u)\, du = o(n^{-r^*/\lambda})$ as $n \to \infty$ for each $0 < \delta \leqslant 1$, where

$$r^* = \begin{cases} r+1 & \text{if } r \text{ is odd,} \\ r & \text{if } r \text{ is even.} \end{cases}$$

(II) For some sequence $\{n_j\}$ of natural numbers and some constant $c_r \neq 0$

$$\lim_{j \to \infty} n_j^{r^*/\lambda} \{G(x) - K_{nr}(G; x)\} = c_r\, g^{(r^*)}(x) \qquad \text{for each } g \in C_0^{r^*+1},$$

where

$$C_0^k = \{g;\, g \in C^k[0, 1],\, g = 0 \text{ on } [0, a] \cup [b, 1]\}.$$

(III) For some $0 < \alpha \leqslant r^*$,

$$\rho_n \int_0^1 u^\alpha k_n(u)\, du = O(n^{-\alpha/\lambda})$$

as $n \to \infty$.

Theorem A. *Assume* I, II, *and* III. *For* $f \in C[I]$ *we have*

(1) $\|f - K_{nr}(F)\|_{C[a,b]} = o(n^{-r^*/\lambda}) \Rightarrow f \in \Pi_{r^*-1}$,

(2) $\|f - K_{nr}(F)\|_{C[a,b]} = O(n^{-r^*/\lambda}) \Rightarrow f^{(r^*-1)} \in \mathrm{Lip}(1; C[I])$,

(3) $f^{(r^*-1)} \in \mathrm{Lip}(1; C[I]) \Rightarrow \|f - K_{nr}(F)\|_{C[a,b]} = O(n^{-r^*/\lambda})$.

From Theorem A we can obtain a characterization of the class $\mathrm{Lip}_r(\alpha; C[I])$ consisting of all functions f with $w_r(f; h) \leqslant O(h^\alpha)$, under the following:

Assumption (IV).

$$\|K_{nr}^{(r)}(F)\|_{C[0,1]} \leqslant M_r \|F\|_{C[0,1]} n^{r/\lambda},$$

where M_r is a constant depending on r.

THEOREM B. *Let* $f \in C[I]$ *and* $0 < \alpha < r$. *If we assume* I, III *and* IV, *we have*

$$\|f - K_{nr}(F)\|_{C[a,b]} = O(n^{-\alpha/\lambda}) \Leftrightarrow f \in \mathrm{Lip}_r(\alpha; C[I]).$$

These theorems are also true for $\| \ \|_{L_p}$, $1 \leqslant p < \infty$, see Section 3.

1. PROOF OF THE THEOREMS

For the proof of Theorem A we need

LEMMA 1.1. *For each* $f \in C[I]$ *consider the extended function* F *constructed by method* (0.2). *Then we have*

$$K_{nr}(F; x) = \rho_n \int_{-1}^{1} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} F(x + ju)\, k_n(u)\, du + o(n^{-r^*/\lambda}),$$

uniformly on the interval $[\delta_0, 1 - \delta_0]$, *and*

$$K_{nr}(F; x) = o(n^{-r^*/\lambda});$$

uniformly on the interval $[0, \delta_0]$ *or* $[1 - \delta_0, 1]$, *as* $n \to \infty$.

Proof. For $\delta_0 \leqslant x \leqslant 1 - \delta_0$ we have $(-x)/j \leqslant (-\delta_0)/r$ and $\delta_0 \leqslant (j - x)/j$. By Assumption I

$$\rho_n \left(\int_{-1}^{-\delta_r} + \int_{\delta_0}^{1} \right) |F(x + ju)|\, k_n(u)\, du \leqslant 2 \|F\|_{C[0,1]} \rho_n \int_{\delta_r}^{1} k_n(u)\, du$$
$$= o(n^{-r^*/\lambda}).$$

Thus we have the first formula.

Let $0 \leqslant x \leqslant \delta_0$ or $1-\delta_0 \leqslant x \leqslant 1$, then we have

$$\left|\int_{(-x)/j}^{(j-x)/j} F(x+ju)\, k_n(u)\, du\right|$$

$$= \left|\left(\int_{-\delta_r}^{\delta_r} + \int_{(-x)/j}^{-\delta_r} + \int_{\delta_r}^{(j-x)/j}\right) F(x+ju)\, k_n(u)\, du\right|$$

$$\leqslant 2\, \|F\|_{C[0,1]} \int_{\delta_r}^{1} k_n(u)\, du.$$

If we use Assumption I again, we have the second formula. Q.E.D.

Proof of Theorem A. (1) Let us assume that

$$\lim_{n\to\infty} n^{r'/\lambda}\{f(x) - K_{nr}(F;x)\} = 0$$

uniformly on $[a,b]$. For each $g \in C_0^{r'+1}$ we have

$$\lim_{n\to\infty} \int_0^1 n^{r'/\lambda}\{F(x) - K_{nr}(F;x)\}\, G(x)\, dx = 0.$$

We put

$$I = \int_{\delta_0}^{1-\delta_0} \{F(x) - K_{nr}(F;x)\}\, G(x)\, dx$$

$$= \int_{\delta_0}^{1-\delta_0} F(x)\, G(x)\, dx - \rho_n \int_{\delta_0}^{1-\delta_0} G(x) \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \left(\frac{1}{j}\right)$$

$$\times \int_0^j F(u)\, k_n\left(\frac{x-u}{j}\right) du\, dx.$$

Then we have

$$I_{nj} = \left(\frac{\rho_n}{j}\right) \int_{\delta_0}^{1-\delta_0} G(x) \int_0^j F(u)\, k_n\left(\frac{x-u}{j}\right) du\, dx$$

$$= \left(\frac{\rho_n}{j}\right) \int_0^j F(u) \int_{\delta_0}^{1-\delta_0} G(x)\, k_n\left(\frac{x-u}{j}\right) dx\, du$$

$$= \rho_n \int_0^j F(u) \int_{(\delta_0-u)/j}^{(1-\delta_0-u)/j} G(u+jt)\, k_n(t)\, dt\, du.$$

Here, $(\delta_0-u)/j \geqslant (\delta_0/j) - 1 \geqslant \delta_r - 1$, $(1-\delta_0-u)/j \leqslant (1-\delta_0)/j \leqslant 1-\delta_0$.

and $(1-\delta_0-u)/j-(\delta_0-u)/j=(1-2\delta_0)/j$. Thus, for u with $(1-\delta_0-u)\leqslant -\delta_r$ or $\delta_r\leqslant(\delta_0-u)/j$, we have

$$\left|\rho_n\int_{(\delta_0-u)/j}^{(1-\delta_0-u)/j} G(u+jt)\,k_n(t)\,dt\right|\leqslant\|G\|_{C[0,1]}\,\rho_n\int_{\delta_r}^{1} k_n(t)\,dt$$
$$=o(n^{-r^*/\lambda}).$$

By the definition of F we have

$$F(u)=0 \qquad \text{on } [\delta_0-j\delta_r,\delta_0+j\delta_r]\cup[1-\delta_0-j\delta_r,1-\delta_0+j\delta_r]. \tag{1.2}$$

Thus we have

$$I_{nj}=\rho_n\int_{\delta_0+j\delta_r}^{1-\delta_0-j\delta_r} F(u)\int_{(\delta_0-u)/j}^{(1-\delta_0-u)/j} G(u+jt)\,k_n(t)\,dt\,du+o(n^{-r^*/\lambda}).$$

But we see $(\delta_0-u)/j\leqslant-\delta_r$ and $(1-\delta_0-u)/j\geqslant\delta_r$. By Assumption I and (1.2) we have

$$I_{nj}=\rho_n\int_{\delta_0+j\delta_r}^{1-\delta_0-j\delta_r} F(u)\int_{(-u)/j}^{(j-u)/j} G(u+jt)\,k_n(t)\,dt\,du$$
$$=\rho_n\int_{\delta_0}^{1-\delta_0} F(u)\int_{(-u)/j}^{(j-u)/j} G(u+jt)\,k_n(t)\,dt\,du+o(n^{-r^*/\lambda}).$$

Consequently, we have

$$I=\int_{\delta_0}^{1-\delta_0} F(u)\{G(u)-K_{nr}(G;u)\}\,du+o(n^{-r^*/\lambda}).$$

By (1.1) and Assumption II,

$$\int_0^1 F(u)\,c_r G^{(r^*)}(u)\,du=0.$$

Thus we have

$$f\in\Pi_{r^*-1} \qquad \text{on } [a,b].$$

(2) From the weak* compactness there are a sequence $\{n_k\}$ of natural numbers and a function $h\in L_\infty[a,b]$ such that

$$\lim_{k\to\infty}\int_0^1 n_k^{r^*/\lambda}\{F(x)-K_{n_k r}(F;x)\}\,G(x)\,dx$$
$$=\int_0^1 h(x)\,G(x)\,dx \qquad \text{for all} \quad g\in C_0^{r^*+1}.$$

In this case, by using same method as the proof of (1) we have

$$\lim_{k\to\infty}\int_0^1 n_k^{r^*/\lambda}F(t)\{G(t)-K_{nr}(G;t)\}\,dt=\int_0^1 h(t)\,G(t)\,dt.$$

By Assumption II,

$$\int_0^1 F(t)\,c_r G^{(r^*)}(t)\,dt=\int_0^1 H_{r^*}(t)\,G^{(r^*)}(t)\,dt,$$

where H_{r^*} is a r^*th integral of h. Thus we have

$$c_rF-H_{r^*}\in\Pi_{r^*-1}\quad\text{on } [a,b]\qquad\text{or}\qquad F^{(r^*)}(t)=c_r^{-1}h(t)\quad\text{a.e.}$$

Consequently, we see

$$f^{(r^*-1)}\in \mathrm{Lip}(1;C[I]).$$

Using Lemma 1.1, for $a\leqslant x\leqslant b$ we have

$$F(x)-K_{nr}(F;x)=\rho_n\int_{-1}^{1}\Delta_u^r F(x)\,k_n(u)\,du+o(n^{-r^*/\lambda}).$$

From $F^{(r^*-1)}\in\mathrm{Lip}(1;C[0,1])$ we see

$$\Delta_u^rF(x)=u^{r^*-1}F^{(r^*-1)}(x)+u^{r^*}h(x,u),$$

where $|h(x,u)|\leqslant M$ for all x, u (see Lemma 2.2, below). By Assumption III,

$$\left|\rho_n\int_{-1}^{1}\Delta_u^rF(x)\,k_n(u)\,du\right|\leqslant M_{rf}\,\rho_n\int_{-1}^{1}u^{r^*}k_n(u)\,du=O(n^{-r^*/\lambda}).$$

Q.E.D.

To complete the proof of Theorem B we need two lemmas. They are well known.

LEMMA 1.2. *Let*

$$F_{\eta r}(x)=(\eta/r)^{-r}\int_{(-\eta)/2r}^{\eta/2r}\cdots\int_{(-\eta)/2r}^{\eta/2r}$$

$$\times\sum_{s=1}^{r}(-1)^{s+1}\binom{r}{j}F(x+s(u_1+\cdots+u_r))\,du_1\cdots du_r,$$

then we have

(1) $|F(x)-F_{\eta r}(x)|\leqslant w_r(F;\eta)$,

(2) *for* $|\eta| \leqslant \delta_r$,

$$F_{\eta r}^{(i)}(x) = 0, \qquad |k - x| \leqslant \delta_0, \qquad i = 0, 1, 2, \ldots, \quad k = 0, \pm 1, \pm 2, \ldots,$$

(3) *for* $|\eta| \leqslant \delta_r$,

$$K_{nr}^{(r)}(F_{\eta r}; x) = K_{nr}(F_{\eta r}^{(r)}; x),$$

and

(4) $|F_{\eta r}^{(r)}(x)| \leqslant M_r \eta^{-r} w_r(F; \eta)$,

where M_r *is a constant depending on* r.

LEMMA 1.3 [5]. *Let* Ω *be monotonely increasing on* $[0, c]$. *Then* $\Omega(t) = O(t^\alpha)$, $t \to 0+$, *if for some* $0 < \alpha < r$ *and all* h, $t \in [0, c]$

$$\Omega(h) \leqslant M\{t^\alpha + (h/t)^r \, \Omega(t)\}.$$

Now, it is easy to show Theorem B.

Proof of Theorem B. ($\Rightarrow$) We use Assumption IV and Lemma 1.2. For $a \leqslant x \leqslant b$

$$\begin{aligned}
|\Delta_h^r f(x)| &\leqslant \|\Delta_h^r \{f - K_{nr}(F; \cdot)\}\| \\
&\quad + \left\| \int_{(-h)/2r}^{h/2r} \cdots \int_{(-h)/2r}^{h/2r} K_{nr}(F; \cdot)(\cdot + u_1 + \cdots + u_r) \, du_1 \cdots du_r \right\| \\
&\leqslant 2^r \|f - K_{nr}(F)\| + (h/r)^r \{\|K_{nr}^{(r)}(F - F_{\eta r}; \cdot)\| + \|K_{nr}^{(r)}(F_{\eta r}; \cdot)\|\} \\
&\leqslant 2^r M_{rf} n^{-\alpha/\lambda} + (h/r)^r \{M_r n^{r/\lambda} \|F - F_{\eta r}\| + M_r' \|F_{\eta r}^{(r)}\|\} \\
&\leqslant 2^r M_{rf} n^{-\alpha/\lambda} + (h/r)^r \{M_r n^{r/\lambda} w_r(F; \eta) + M_r' \eta^{-r} w_r(F; \eta)\} \\
&\leqslant M_{rf}' \{n^{-\alpha/\lambda} + (n^{1/\lambda} h)^r \, w_r(F; n^{-1/\lambda})\}
\end{aligned}$$

with $\eta = n^{-1/\lambda}$. Thus, for $0 < t \leqslant 1$ we have

$$w_r(f; h) \leqslant M_{rf}'' \{t^\alpha + (h/t)^r \, w_r(f; t)\}.$$

From Lemma 1.3,

$$w_r(f; h) = O(h^\alpha).$$

($\Leftarrow$) By Lemma 1.1 and Assumption III,

$$\begin{aligned}
F(x) - K_{nr}(F; x) &= \rho_n \int_{-1}^{1} \Delta_u^r F(x) \, k_n(u) \, du + o(n^{-r^*/\lambda}) \\
&= O(1) \int_0^1 u^\alpha k_n(u) \, du + o(n^{-r^*/\lambda}) = O(n^{-\alpha/\lambda}). \qquad \text{Q.E.D.}
\end{aligned}$$

2. Applications

As mentioned in Section 0, our theorems are applicable to Jackson- or Korovkin-type operators. First, we give concrete examples of Korovkin-type operators.

EXAMPLES 1. Let $\eta > 0$.

$$\varphi(t) = \psi(t^\eta) \qquad \text{for} \quad t \geqslant 0.$$

(1) $\varphi(t) = e^{-|t|^\eta}$.
Weierstrass kernel [6] if $\eta = 2$,
Picard kernel if $\eta = 1$,
Bui, Fedorov, Cervakov kernel [7] if $\eta = 1/k$, $k = 1, 2, \ldots$.

(2) $\varphi(t) = 1 - t^\eta$, $\eta > 0$.
Landau kernel [8] if $\eta = 2$,
Mamedov kernel [9] if $\eta = 2k$, $k = 1, 2, \ldots$.

(3) $\varphi(t) = 1/(1 + \sum_{k=1}^{\infty} c_k t^{\eta k})$, where the all coefficients c_k are positive.
Mirakian kernel [10] if $\eta = 2$ and $c_k = 1/2^{2k+1}k$, $k = 1, 2, \ldots$.

EXAMPLE 2. $\varphi(t) = \cos^\eta(t/2)$.

de la Vallee–Poussin kernel [11] if $\eta = 2$.

The following lemmas are fundamental.

LEMMA 2.1. *If $g \in C^{r+1}$ we have*

$$\Delta_u^r g(x) = \begin{cases} a_r u^r g^{(r)}(x) + a_{r+1} u^{r+1} g^{(r+1)}(x) \\ \qquad + a_{r+2} u^{r+2}\{g^{(r+2)}(x) + h(x,u)\} & \text{if } r \text{ is odd}, \\ b_r u^r g^{(r)}(x) + b_{r+1} u^{r+1}\{g^{(r+1)}(x) + h(x,u)\} & \text{if } r \text{ is even}. \end{cases}$$

Here, $h(x, u)$ is a continuous function and

$$\lim_{u \to 0} h(x,u) = 0 \qquad \textit{uniformly in } x,$$

and

$$(r+p)!\, a_{r+p} = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^{r+p}, \qquad p = 0, 1, 2,$$

$$(r+q)!\, b_{r+q} = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^{r+q}, \qquad q = 0, 1.$$

Proof. When r is odd we put

$$h(x,u)=\begin{cases}(a_{r+2}u^{r+2})^{-1}\{\Delta_u^r g(x)-a_r u^r g^{(r)}(x)-a_{r+1}u^{r+1}g^{(r+1)}(x) \\ \qquad\qquad -a_{r+2}u^{r+2}g^{(r+2)}(x)\} & \text{if } u\neq 0, \\ 0 \quad \text{if } u=0.\end{cases}$$

In this case, from the Cauchy theorem

$$\begin{aligned}\lim_{u\to 0} h(x,u) &= \lim_{u\to 0}\{(r+2)!\,a_{r+2}\}^{-1} \\ &\quad\times\left\{\sum_{j=0}^{r}(-1)^{r-j}j^{r+2}g^{(r+2)}(x+ju)-(r+2)!\,a_{r+2}\,g^{(r+2)}(x)\right\} \\ &=\{(r+2)!\,a_{r+2}\}^{-1}g^{(r+2)}(x) \\ &\quad\times\left\{\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}j^{r+2}-(r+2)!\,a_{r+2}\right\} \\ &=0.\end{aligned}$$

Similarly, for even r we obtain the second equality. Q.E.D.

LEMMA 2.2. *Let* $0<\alpha$.

(1) *If* $[\alpha]\neq\alpha$, *for* g *with* $g^{([\alpha])}\in \mathrm{Lip}(\alpha-[\alpha]; C[0,1])$ *we have*

$$\Delta_u^{[\alpha]}g(x)=u^{[\alpha]}g^{([\alpha])}(x)+u^{\alpha}h(x,u).$$

(2) *If* $[\alpha]=\alpha$, *for* g *with* $g^{(\alpha-1)}\in \mathrm{Lip}(1; C[0,1])$ *we have*

$$\Delta_u^{\alpha-1}g(x)=u^{\alpha-1}g^{(\alpha-1)}(x)+u^{\alpha}h(x,u).$$

Here, $h(x,u)$ *is continuous except for* $u=0$ *and uniformly bounded;* $|h(x,u)|\leqslant M$ *for all* x, u.

Proof. (1) We put

$$h(x,u)=\begin{cases}u^{-\alpha}\{\Delta_u^{[\alpha]}g(x)-u^{[\alpha]}g^{([\alpha])}(x)\} & \text{if } u=0 \\ 0 & \text{if } u=0.\end{cases}$$

For $u\neq 0$, we see

$$\begin{aligned}|h(x,u)| &= |u^{-\alpha}\{\Delta_u^{[\alpha]}g(x)-u^{[\alpha]}g^{([\alpha])}(x)\}| \\ &\leqslant u^{-\alpha}\int_0^u\cdots\int_0^u |g^{([\alpha])}(x+u_1+\cdots+u_{[\alpha]})-g^{([\alpha])}(x)| \\ &\qquad\times du_1\cdots du_{[\alpha]}\end{aligned}$$

$$\leqslant M_1 u^{-\alpha} \int_0^u \cdots \int_0^u (u_1 + \cdots + u_{[\alpha]})^{\alpha-[\alpha]}\, du_1 \cdots du_{[\alpha]}$$

$$\leqslant M_1 u^{-\alpha}([\alpha] u)^{\alpha-[\alpha]} u^{[\alpha]}$$

$$\leqslant M_1 [\alpha] \equiv M,$$

where M_1 is a constant.

(2) This follows by the same method as (1). Q.E.D.

The next lemma follows by [1, Chap. 4, Sect. 3].

LEMMA 2.3.

(1)
$$\lambda^{-1}_{[n'/2]} \approx n^{-2p+1} = \begin{cases} n^{-(r+2)} & \textit{if} \quad r \textit{ is odd} \\ n^{-(r+1)} & \textit{if} \quad r \textit{ is even}, \end{cases}$$

where $n' = [n/p] + 1$.

(2) *For* $0 \leqslant \alpha \leqslant 2p-2$ *we have*

$$\rho_n \int_0^1 u^\alpha k_{np}(u)\, du \approx n^{-\alpha}.$$

If we use the three lemmas mentioned above, we obtain the main lemma with respect to the kernel (0.1).

LEMMA 2.4. *The kernel* k_{np} *satisfies Conditions* I ~ IV.

Proof. Let $\lambda = 1$.

(I) By Lemma 2.3(1),

$$\rho_n \int_\delta^1 k_{np}(u)\, du = \rho_n \int_{-1}^{-\delta} k_{np}(u)\, du \approx \lambda^{-1}_{[n'/2]} = o(n^{-r^*}).$$

(II) By Lemma 2.3(2),

$$\rho_n \int_0^1 u^k k_{np}(u)\, du \approx n^{-k}, \qquad k = 0, 1, \ldots, 2p-2.$$

From

$$r^* = \begin{cases} r + 1 = 2p - 2 & \text{if} \quad r \text{ is odd} \\ r = 2p - 2 & \text{if} \quad r \text{ is even}, \end{cases}$$

there is a sequence $\{n_j\}$ of natural numbers and a constant $c'_r \neq 0$ depending on r such that

$$\lim_{j\to\infty} n_j^{r^*}\rho_{n_j}\int_0^1 u^{r^*}k_{n_jp}(u)\,du = c'_r.$$

Let r be odd. By Lemma 2.1, for $g \in C_0^{r^*+1}$ we have

$$\begin{aligned}
&\lim_{j\to\infty} n_j^{r^*}\{G(x) - I_{n_jpr}(G;x)\}\\
&\quad= \lim_{j\to\infty} n_j^{r+1}\rho_{n_j}\int_0^1 \Delta_u^r G(x)\,k_{n_jp}(u)\,du\\
&\quad= \lim_{j\to\infty} n_j^{r+1}\rho_{n_j}\int_{-1}^1 \{a_{r+1}u^{r+1}g^{(r+1)}(x) + a_{r+2}u^{r+2}h(x,u)\}\,k_{n_jp}(u)\,du\\
&\quad= a_{r+2}c'_r\,g^{(r+1)}(x) + I_2.
\end{aligned}$$

But, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\begin{aligned}
|I_2| &\leqslant 2\varepsilon a_{r+2}\lim_{j\to\infty} n_j^{r+1}\rho_{n_j}\int_0^{\delta_u r+1} k_{n_jp}(u)\,du\\
&\quad+ 2a_{r+2}\sup_{x,u}|h(x,u)|\lim_{j\to\infty} n_j^{r+1}\rho_{n_j}\int_\delta^1 k_{n_jp}(u)\,du\\
&\leqslant 2(a_{r+2}c'_r + 1)\varepsilon.
\end{aligned}$$

Thus, $I_2 = 0$ and we have

$$\lim_{j\to\infty} n_j^{r^*}\{G(x) - I_{n_jpr}(G;x)\} = c_r\,g^{(r^*)}(x).$$

This also follows when r is even.

(III) This follows by Lemma 2.3(2).

(IV) Inductively, the following is shown:

$$\left(\frac{d^r}{du^r}\right)k_{np}(u) = 2\{\cos(t/2)\}^{-(2r-1)}\sum_{i=1}^{r} T_{ri}(t)\,K_{[n/2]p}^{(i)}(t),$$

where $u = \sin(t/2)$ and

$$T_{ri}(t) = \begin{cases} \text{a polynomial of degree } r-1 \text{ or less} & \\ \qquad\qquad\text{with valuable } \sin(t/2) & \text{if } r \text{ is odd},\\ \{\cos(t/2)\}\cdot\{\text{a polynomial of degree } (r-2) \text{ or less} & \\ \qquad\qquad\text{with valuable } \sin(t/2)\} & \text{if } r \text{ is even}.\end{cases}$$

For $0 \leqslant x \leqslant 1$ and $\delta_0 \leqslant u \leqslant j - \delta_0$ we see $(x-u)/j \leqslant 1 - \delta_0$ and $(x-u)/j \geqslant -1 + \delta_r$. In this case we have

$$\left| \left(\frac{d^r}{dx^r}\right)\left(\frac{\rho_n}{j}\right) \int_{\delta_0}^{j-\delta_0} F(u)\, k_{np}\left(\frac{x-u}{j}\right) du \right| \leqslant M_{rj} \|F\|_{C[0,1]}\, n^r.$$

Thus we have

$$|I_{npr}^{(r)}(F; x)| \leqslant M_r \|F\|_{C[0,1]}\, n^r. \qquad \text{Q.E.D.}$$

Thus we obtain the following results.

COROLLARY 2.1. *For $f \in C[a, b]$ we have*

(1) $\|f - I_{npr}(F)\|_{C[a,b]} = o(n^{-r'}) \Rightarrow f \in \Pi_{r'-1}$.

(2) $\|f - I_{npr}(F)\|_{C[a,b]} = O(n^{-r'}) \Leftrightarrow f^{(r'-1)} \in \mathrm{Lip}(1; C[I])$.

COROLLARY 2.2. *Let $f \in C[a, b]$ and $0 < \alpha < r$. Then we have*

$$\|f - I_{npr}(F)\|_{C[a,b]} = O(n^{-\alpha}) \Leftrightarrow f \in \mathrm{Lip}_r(\alpha; C[I]).$$

In the next place we consider the application to a Korovkin-type operator.

LEMMA 2.5 [3]. *Let φ be a non-negative and decreasing function on $[0, c]$, $\varphi(0) = 1$, $0 \leqslant \varphi(x) < 1$ if $0 < x \leqslant c$, and*

$$\lim_{x \to 0+} \{1 - \varphi(x)\}/x^\lambda = d, \tag{2.2}$$

where λ and d are positive numbers. Then, for every $\beta \geqslant 0$ and $n = 1, 2, \ldots,$ we have

$$\begin{aligned} &B(\lambda,\beta)(nd)^{-(\beta+1)/\lambda} - (2d)^{-(\beta+1)/\lambda} e^{-2nd\eta^\lambda} \\ &\leqslant \int_0^1 t^\beta \varphi^n(t)\, dt \\ &\leqslant A(\lambda,\beta)(nd)^{-(\beta+1)/\lambda} + e^{-nd\eta^\lambda/2}, \end{aligned}$$

where $A(\lambda, \beta)$ and $B(\lambda, \beta)$ are positive constants depending on λ and β, and η is a certain positive constant. Thus,

$$\left\{\int_0^1 t^\beta \varphi^n(t)\, dt\right\} \Big/ \left\{\int_0^1 \varphi^n(t)\, dt\right\} \approx n^{-\beta/\lambda}.$$

LEMMA 2.6. *Let φ satisfy condition* (2.2) *and the condition*

$$|(d/du)\,\varphi(u)| \leqslant Mu^{\lambda-1}, \qquad 0 \leqslant u \leqslant 1, \tag{2.3}$$

where M is a constant. Then condition IV *is satisfied.*

Proof. We use the lemma 2.5.

$$\begin{aligned}
&\left|\left(\frac{1}{j}\right)\rho_n \int_0^j F(u)\left(\frac{\partial^r}{\partial x^r}\right)\varphi^n\left(\frac{x-u}{j}\right)du\right| \\
&= \left|\left(\frac{-1}{j}\right)^r \rho_n \int_{(-x)/j}^{(j-x)/j} F(x+ju)\left(\frac{d^r}{du^r}\right)\varphi^n(u)\,du\right| \\
&\leqslant 2j^{-r}\,\|F\|_{C[0,1]}\,\rho_n \int_0^1 n^P r\varphi^{n-r}(u)\left|\left(\frac{d}{du}\right)\varphi(u)\right|^r du \\
&\leqslant 2Mj^{-r}\,\|F\|_{C[0,1]}\,n^P r\,\rho_n \int_0^1 u^{r(\lambda-1)}\varphi^{n-r}(u)\,du \\
&\leqslant M_{rj}\,\|F\|_{C[0,1]}\,n^{r/\lambda}.
\end{aligned}$$

Q.E.D.

Condition (2.2) is satisfied for all of the examples mentioned in the beginning of this section. Further, if we put $\eta = 2k$, $k = 1, 2, \ldots$, in the examples we see that they satisfy condition (2.3), too. For the operators of this kind we obtain the following main lemma.

LEMMA 2.7. *If a Korovkin-type operator satisfies condition* (2.2), *it satisfies also Conditions* I–III. *Further, if that operator also satisfies* (2.3) *it satisfies Condition* IV.

Proof. (I) For $0 \leqslant \delta_1 < 1$,

$$\begin{aligned}
\rho_n \int_\delta^1 k_n(u)\,du = \rho_n \int_{-1}^{-\delta} k_n(u)\,du &\approx O(\rho_n \varphi^n(\delta)) \\
&= O(1)\{2/(1-\delta_1)\}\{\varphi(\delta)/\varphi(\delta_1)\}^n.
\end{aligned}$$

(II) Using Lemma 2.1 and 2.5 we see that Condition II is satisfied. Its proof is same as Lemma 2.4(II).

(III) This follows from Lemma 2.5.

(IV) This follows from Lemma 2.6. Q.E.D.

By Lemma 2.7 we have

COROLLARY 2.3. *Let a Korovkin-type kernel* (0.4) *satisfy Condition* (2.2). *Then, for* $f \in C[a, b]$ *and the operator* (0.5) *we have*

(1) $\|f - K_{nr}(F)\|_{C[a,b]} = o(n^{-r/\lambda}) \Rightarrow f \in \Pi_{r-1}$.

(2) $\|f - K_{nr}(F)\|_{C[a,b]} = O(n^{-r/\lambda}) \Leftrightarrow f^{(r-1)} \in \mathrm{Lip}(1; C[I])$.

COROLLARY 2.4. *Let a Korovkin-type kernel* (0.4) *satisfy Condition* (2.2) *and* (2.3). *Then, for* $f \in C[a, b]$ *and the operator* (0.5) *we obtain that for* $0 < \alpha < r$

$$\|f - K_{nr}(F)\|_{C[a,b]} = O(n^{-\alpha/\lambda}) \Leftrightarrow f \in \mathrm{Lip}_r(\alpha; C[I]).$$

3. L_p, $1 \leqslant p < \infty$, CASE.

For each $f \in L_p[I]$ we consider a function $F \in L_p[0, 1]$, which is extended by the same methods as the uniform case. However, we have to change Condition (4) in (0.2) by the condition; (4'). F is sufficiently smooth, for instance, if $f(x) = H(x)$ a.e. on $[a, b]$, where $H^{(k)} \in \mathrm{Lip}(1; L_p[a, b])$, there is a function H_1 such that $F(x) = H_1(x)$ a.e. on $[0, 1]$, and $H_1^{(k)} \in \mathrm{Lip}(1; L_p[0, 1])$, or there is a constant M_{rf} depending on r and f such that $w_{rp}(F; h) \leqslant M_{rf} w_{rf}(f; h)$, $0 < h \leqslant (b - a)/r$, where $w_{rp}(g; \cdot)$ is the integral modulus of smoothness of order r for g. Of course, we consider the L_p-norm through this section.

By the same lines as the uniform case we obtain the following theorems. The proofs are omitted.

THEOREM C. *If we assume* I, II *and* III, *for* $f \in L_p[I]$ *we have*

(1) $\|f - K_{nr}(F)\|_{L_p[a,b]} = o(n^{-r/\lambda}) \Rightarrow \exists P \in \Pi_{r-1} : f(x) = P(x)$ a.e..

(2) $\|f - K_{nr}(F)\|_{L_p[a,b]} = O(n^{-r/\lambda}) \Leftrightarrow \exists H \in L_p^{r-1}[0, 1]$: $f(x) = H(x)$ a.e., $H^{(r-1)} \in \mathrm{Lip}(1; L_p[0, 1])$,

where $g \in L_p^k[0, 1]$ *means* $g^{(k)} \in L_p[0, 1]$.

THEOREM D. *Let* $f \in L_p[I]$ *and* $0 < \alpha < r$. *If we assume* I, III *and* IV, *where we change the uniform-norm in* IV *by the* L_p-*norm, we have*

$$\|f - K_{nr}(F)\|_{L_p[a,b]} = O(n^{-\alpha/\lambda}) \Leftrightarrow f \in \mathrm{Lip}_r(\alpha; L_p[a, b]).$$

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